ESE 4310 – L#2 Mathematical Foundations

- Functions =
  - Exponential / Exponents ✓ $n!$, 2^n
  - Logarithms ✓ $\log_2 n$
  - ✓ Ceiling / Floor
  - Code Analysis ← ✓ Summation
  - ✓ Lopital's limit Rule. — [$O(\cdot)$, $\Omega(\cdot)$, $\Theta(\cdot)$]
  - ✓ Derivatives

- Exponents. X^A, X^B $\Rightarrow$

$$X^A \cdot X^B = X^{A+B}$$

Ex. $2^3 \cdot 2^4 = 2^7$

$$X^A / X^B = X^{A-B}$$

$$\left[\frac{1}{X} = X^{-1} \right]$$

$$(X^A)^B = X^{AB}$$

$$\frac{1}{X^B} = X^{-B}$$

$$X^n + X^n = \begin{cases} 2X^n \checkmark \\ \times\, X^{2n} \equiv X^n \cdot X^n \end{cases}$$

$$X^{A/B} = \sqrt[B]{X^A} = \left(\sqrt[B]{X}\right)^A$$

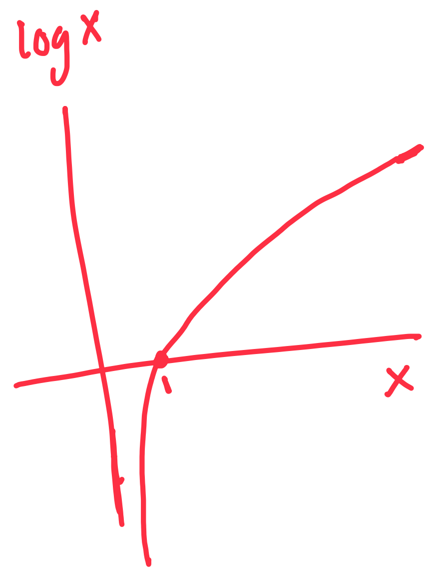
Ex. $X^{2/3} = \sqrt[3]{X^2} = \left(\sqrt[3]{X}\right)^2$

• Logarithm :

$\log_a x \leftarrow$ base

usually in most CS

base = $\underline{\underline{2}}$



$$y = \log_a x \Rightarrow x = a^y$$

regardless of the $\underline{\text{base}}$:

$$\log AB = \log A + \log B$$

$$\log A/B = \log A - \log B$$

$$\log (A)^B = B \log A$$

$\underline{\underline{\text{Ex}}}$ $\log_{10} x^2 = 2 \log_{10} x$

<u>important property of logarithm:</u>

$A, B, C > 0$

(1) $$\log_A B = \frac{\log_c B}{\log_c A}$$

convert between bases.

(2) $$X_1^{\log_b X_2} = X_2^{\log_b X_1}$$

$\underline{\underline{\text{Ex}}}$ $$3^{\log_3 n} = n^{\log_3 3} = n$$

$$4^{\log_3 n} = n^{\log_3 4} \cong n^*$$

* Assume the default base = $\underline{\underline{2}}$ unless otherwise specified

$2\log n \Rightarrow 2 \cdot \log_{\underline{\underline{2}}} n$

what is $B^{\log_B N} = N = N^{\log_B B} = N^1 = N$ ✓

- $\ln x = \log_e x$ Natural Logarithmic function

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{\ln(g(x))}$$

- $f(x) = b^x$ $b>0$ what $f'(x)$?

$f'(x) = b^x \cdot \ln b$

$$f(x) = b^{g(x)} \Rightarrow f'(x) = b^{g(x)} \cdot g'(x) \cdot \ln b$$

- $f(x) = \log_b g(x) \Rightarrow f'(x) =$

$f(x) = \underline{\underline{\log_b x}} \Rightarrow f'(x) = \dfrac{1}{x \ln b}$

let $y = \log_b x \Rightarrow b^y = b^{\log_b x} = x$

$b^y = x$

$\ln(b^y) = \ln(x)$

$$y \ln(b) = \ln(x)$$

$$y = \frac{\ln(x)}{\ln b}$$

$$y' = \frac{\frac{1}{x}}{\ln b} = \frac{1}{x \ln b}$$

$$f(x) = \log_b g(x) \Rightarrow f'(x) = \frac{g'(x)}{g(x) \cdot \ln(b)}$$

- Lopital's rule

$$\lim_{x \to c} \frac{f}{g} = \frac{\infty}{\infty}, \frac{0}{0}, 0 \times \infty, 1^{\infty}, 0^{0}, \infty - \infty$$

Indeterminant limit

$$\lim_{x \to c} \frac{f'}{g'} = \frac{\infty}{\infty} \cdots$$

$$\lim_{x \to c} \frac{f''}{g''} \cdots$$

Ex. $\lim_{x \to 0} \frac{3^x - 2^x}{x^2 - x} = \frac{3^0 - 2^0}{0 - 0} = \frac{1 - 1}{0 - 0} = \frac{0}{0}$

$\Rightarrow$ Lopital's Rule! $\lim_{x \to 0} \frac{3^x - 2^x}{x^2 - x} = \lim_{x \to 0} \frac{3^x \cdot \ln(3) - 2^x \cdot \ln(2)}{2x - 1}$

$$= \frac{3^0 \cdot \ln(3) - 2^0 \cdot \ln(2)}{-1}$$

$$= \ln 2 - \ln 3$$

Ex Find $\lim_{n\to\infty} \frac{n^2}{n \cdot \log_2 n} = \frac{\infty}{\infty} =$

$= \lim_{n\to\infty} \frac{n}{\log_2 n} = \frac{\infty}{\infty}$

Lopitals Rule $\leftarrow$ LR

$= \lim_{n\to\infty} \frac{1}{\frac{1}{n \ln 2}} = \lim_{n\to\infty} n \ln 2 = \infty$

$\Rightarrow = \lim_{n\to\infty} \frac{n}{\log_2 n} = \frac{\infty}{\infty}$

Apply Lopital's $\Rightarrow \lim_{n\to\infty} \frac{1}{\frac{1}{n \ln(2)}}$

$= \lim_{n\to\infty} \frac{n \cdot \ln(2)}{1}$

$= \frac{\infty \cdot \ln(2)}{1} = \underline{\underline{\infty}}$.

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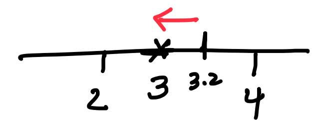
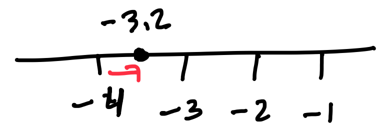
- Floor $\lfloor x \rfloor$ = greatest Integer less than or equal to x.

Ex.

$\lfloor 3.2 \rfloor = 3$



$\lfloor -3.2 \rfloor = -4$



$x-1 < \lfloor x \rfloor \leq x$

- Ceiling function : Least Integer greater than or equal to x.

$x \leq \lceil x \rceil < x+1$

Ex

$\lceil 3.2 \rceil = 4$

$\lceil -3.2 \rceil = -3$

- Series : $\# \displaystyle\sum_{i=0}^{N} 2^i = 2^{N+1} - 1$ ✓ ✗ special for base = 2

Ex. $\displaystyle\sum_{i=0}^{4} 2^i = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1+2+4+8+16$
$= \underline{\underline{31}}$

$\displaystyle\sum_{i=0}^{4} 2^i = 2^5 - 1 = 32 - 1 = 31$ ✓

• $\sum_{i=0}^{7}(2*3^i + 5*2^i) = \sum_{i=0}^{7} 2.3^i + \sum_{i=0}^{7} 5.2^i$

$= 2.\sum_{i=0}^{7} 3^i + 5*\sum_{i=0}^{7} 2^i =$

$5*(2^8-1)$

1275

general

$$\sum_{i=0}^{N} a*r^i = \begin{cases} \dfrac{a*r^{N+1} - a}{r-1} & r \neq 1 \\ (N+1)a & r = 1 \end{cases}$$

$$\sum_{i=0}^{7} 2\cdot 3^i \Rightarrow \begin{array}{l} a=2 \\ r=3 \end{array} \Rightarrow \frac{2(3)^8-2}{2}$$

$= 6560$

$$\Rightarrow \sum_{i=0}^{7}\left(2*3^i + 5*2^i\right) = 6560+1275$$

$= \boxed{7835}$

$$\sum_{i=0}^{7} 2*3^i = 2*\left[3^0+3^1+3^2+3^3+3^4+3^5+3^6+3^7\right] = 6560$$

$+$

$$\sum_{i=0}^{7} 5*2^i = 5*\left[2^0+2^1+ - - - - +2^7\right] = 1275$$

$$\sum_{i=0}^{7} 5*2^i \Rightarrow \begin{array}{l} a=5 \\ r=2 \end{array} \Rightarrow \frac{5(2)^8-5}{1}$$

$= 5[2^8-1]$

- $$\sum_{i=0}^{N} i = 0+1+2+3+4 \cdots n = \frac{N(N+1)}{2}$$

- $$\sum_{i=0}^{N} i^2 = 0+1+4+9+16+\cdots n^2 = \frac{N(N+1)(2N+1)}{6}$$

- $$\sum_{i=1}^{N} \frac{1}{i} = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \simeq \ln n \simeq \log_e n$$

- Gauss Identity $\Rightarrow$ $$\sum_{i=0}^{n-1} i = 0+1+\cdots(n-1) = \frac{n(n-1)}{2}$$

- $$\sum_{i=0}^{n-1} i^2 = \frac{n(n-1)(2n-1)}{6}$$

Finite geometric series:

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1} \quad \cdots\cdots$$

$\underline{\underline{EX}}$. $$\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} 1 = ?$$

$\underline{\text{hint:}}$ $$\sum_{i=0}^{n} 1 = 1+1+1+\cdots 1$$
$$= [n-0]+1 = n+1$$

$$\sum_{i=3}^{n+1} 1 = [(n+1)-(3)]+1 = (n+1-3)+1$$
$$= n-1$$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} 1 = \sum_{i=0}^{n-1} 1\cdot((i-1)-0+1) = \sum_{i=0}^{n-1} i$$

using Gauss identity

$$\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2} \text{ or } \frac{n^2}{2} - \frac{n}{2} \checkmark$$